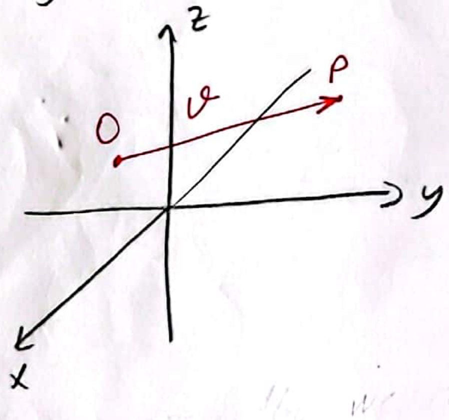
Vectors in Space, n-Vectors

In vector calculus classes, you encountered three-dimensional vectors. Now we will develop the notion of n-vectors and learn some of their properties.

We begin by looking at the space R^n, which we can think of as the space of points with n-coordinates. We then specify an origin O, a favorite point in R^n. Now given any other point P, we can draw a vector v from O to P. Just as in R^3, a vector has a magnitude and a direction.

If O has coordinates $(o^1, \dots o^n)$ and p has coordinates $(p^1, \dots p^n)$ then the components of the vector v are $\begin{pmatrix} p^1 - o^1 \\ p^2 - o^2 \\ \vdots \\ p^n - o^n \end{pmatrix}$. This construction allows us to put the origin anywhere that seems most convinient in R^n, not just at the point with zero coordinates.

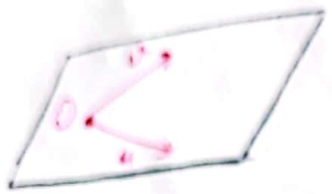


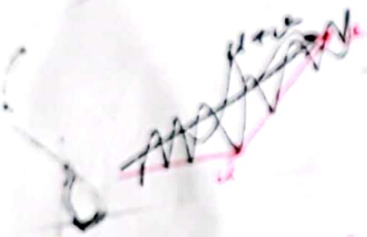
The line determined by a non-zero vector v through a point p can be written as $\{p + tv \mid t \in R\}$. For example

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| \; t \in R \right\}$$

describes a line in 4-dimensional space parallel to the x-axis.

Given two non-zero vectors u, v, they will usually determine a plane, unless both vectors are in the same line.





Directions and Magnitudes of Vectors

The magnitude of a vector is the distance from the endpoint of the vector to the origin. Suppose we want to calculate the magnitude of the vector $\vec{a} = [4, 3]$, which extends 4 units along the x-axis, and 3 units along the y-axis. To calculate the magnitude $\vec{a}$, we can use the Pythagorean theorem.

$$\|\vec{a}\| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$$

consider the Euclidean length of a vector:

$$\|v\| = \sqrt{(v^1)^2 + (v^2)^2 + \cdots (v^n)^2} = \sqrt{\sum_{i=1}^{n} (v^i)^2}$$

A unit vector with a magnitude of '1' is denoted by a hat above the symbol (for instance, $\hat{a}$). $\hat{a}$ is calculated by dividing the original vector ($\vec{a}$) with its magnitude ($\|\vec{a}\|$)

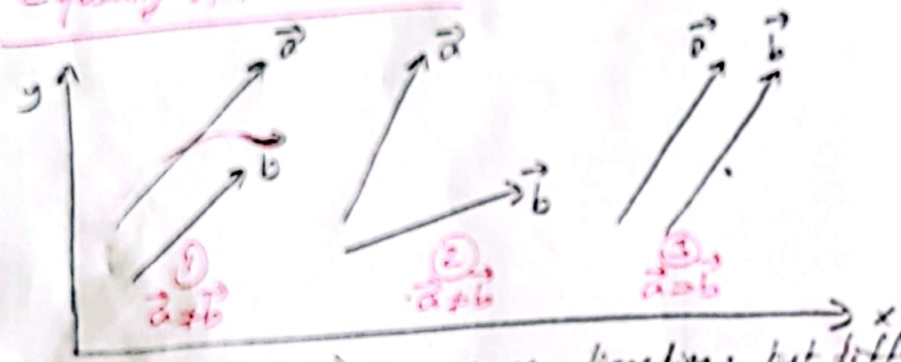
$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

$$\hat{a} = \frac{[4,3]}{5} \Rightarrow \hat{a} = \left[\frac{4}{5}, \frac{3}{5}\right]$$

The magnitude of the unit vector must be equal to 1.

$$\|\hat{a}\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

Equality of two vectors



Exp1: $\vec{a}$ and $\vec{b}$ have same directions but different magnitudes.

Exp2: $\vec{a}$ and $\vec{b}$ have same magnitudes but different directions.

Exp3: $\vec{a}$ and $\vec{b}$ " " " and directions.

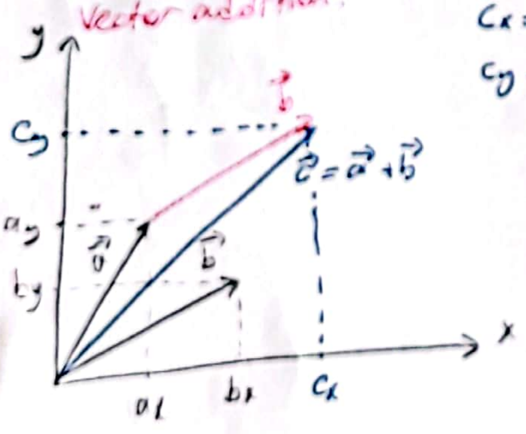
Vector addition and subtraction

Vector addition:

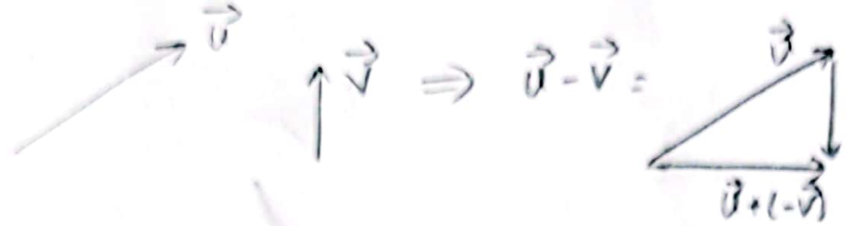
$c_x = a_x + b_x$

$c_y = a_y + b_y$

The sum of $\vec{a}$ and $\vec{b}$ corresponds to laying two vectors head-to-tail and drawing the connecting vector.



Vector subtraction:

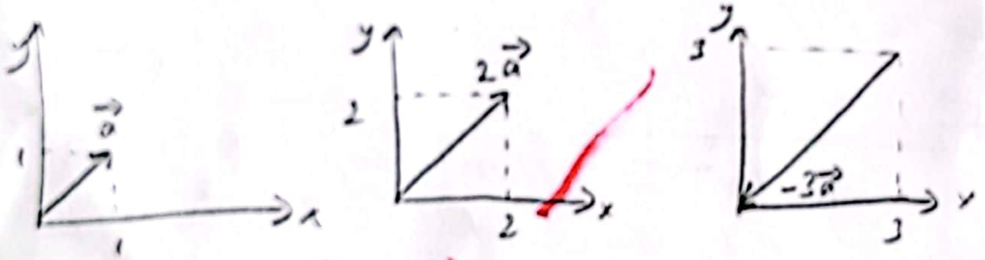


$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

1- Switch the direction of the vector that is being subtracted.
2- Arrange the two vectors from head to tail
3- Draw a resultant vector from the the tail of the first vector to the head of the second.

Multiplying a vector by a scalar

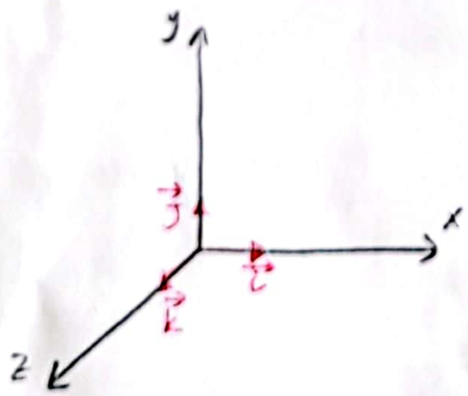
Suppose we have a vector $\vec{a}$, then if this vector is multiplied by a scalar quantity k, then we get a new vector with magnitude as $|k\vec{a}|$ and the direction remains same as the vector $\vec{a}$ if k is positive and if k is negative then the direction of k becomes just opposite of the direction of vector $\vec{a}$.



Unit vectors: $\vec{i}, \vec{j}, \vec{k}$

$\vec{i}, \vec{j}$ and $\vec{k}$ vectors are unit vectors in the direction of x, y and z, respectively

$\vec{i} = (1, 0, 0)$ $\quad \vec{j} = (0, 1, 0)$ $\quad \vec{k} = (0, 0, 1)$



Examples:

1) Find the magnitude of given vector $\vec{a}_1 = 3\vec{i} - 2\vec{j} + \vec{k}$

$$\|\vec{a}_1\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

2) Find the magnitude of $\vec{a}_1 + \vec{a}_2 + \vec{a}_3$

$\vec{a}_1 = 3\vec{i} - 2\vec{j} + \vec{k}$ $\quad \vec{a}_2 = 2\vec{i} - 4\vec{j} - 3\vec{k}$ $\quad \vec{a}_3 = -\vec{i} + 2\vec{j} + 2\vec{k}$!

$$\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = (3\vec{i} - 2\vec{j} + \vec{k}) + (2\vec{i} - 4\vec{j} - 3\vec{k}) + (-\vec{i} + 2\vec{j} + 2\vec{k})$$

$$= 4\vec{i} - 4\vec{j} + 0\vec{k} = 4\vec{i} - 4\vec{j}$$

$$\|\vec{a}_1 + \vec{a}_2 + \vec{a}_3\| = \sqrt{4^2 + (-4)^2 + 0^2} = \sqrt{32} = 4\sqrt{2}$$

3) Find the magnitude of resultant vector $\vec{R} = \vec{a}_1 + \vec{a}_3$, and unit vector of resultant vector.

$\vec{a}_1 = 2\vec{i} + 4\vec{j} - 5\vec{k}$

$\vec{a}_3 = \vec{i} + 2\vec{j} + 3\vec{k}$

$$\vec{R} = (2\vec{i} + 4\vec{j} - 5\vec{k}) + (\vec{i} + 2\vec{j} + 3\vec{k}) = 3\vec{i} + 6\vec{j} - 2\vec{k}$$

$$\|\vec{R}\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{49} = 7$$

$$\hat{R} = \frac{\vec{R}}{\|\vec{R}\|} = \frac{3\vec{i} + 6\vec{j} - 2\vec{k}}{7} = \left(\frac{3}{7}\right)\vec{i} + \left(\frac{6}{7}\right)\vec{j} - \left(\frac{2}{7}\right)\vec{k}$$

Some rules for scalar multiplication of two vectors

1- $\vec{a}.\vec{b} = \vec{b}.a$

2- $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

3- $m(\vec{a}.\vec{b}) = (m\vec{a})\vec{b} = \vec{a}(m\vec{b}) = (\vec{a}.\vec{b})m$

4- $\vec{i}\vec{i} = \vec{j}\vec{j} = \vec{k}.\vec{k} = 1$, $\quad \vec{i}\vec{j} = \vec{j}.\vec{k} = \vec{k}\vec{i} = 0$

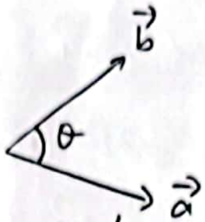
5- $\left.\begin{array}{l} \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \\ \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}. \end{array}\right\}$ $\quad \vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\vec{a}.\vec{a} = a_1^2 + a_2^2 + a_3^2$$

$$\vec{b}.\vec{b} = b_1^2 + b_2^2 + b_3^2$$

The scalar multiplication of two vectors, $\vec{a}$ and $\vec{b}$ is equal to multiplying of the magnitude of these vectors a and b and the cos of angle between the vectors:

$$\vec{a}.\vec{b} = \|\vec{a}\|.\|\vec{b}\|.\cos\theta$$



4) Find the angle between the given vectors

$\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ $\quad \vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$

$$\|\vec{a}\|.\|\vec{b}\| = \left(\sqrt{2^2+2^2+(-1)^2}\right)\left(\sqrt{6^2+(-3)^2+2^2}\right) = \sqrt{9}.\sqrt{49} = 3.7 = 21$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\|\vec{a}\|.\|\vec{b}\|} = \frac{(2\vec{i}+2\vec{j}-\vec{k})(6\vec{i}-3\vec{j}+2\vec{k})}{21}$$

$$= \frac{12\vec{i}\vec{i} \cancel{-6\vec{i}\vec{j}}^{0} \cancel{+6\vec{i}\vec{k}}^{0} \cancel{+12\vec{j}\vec{i}}^{0} - 6\vec{j}\vec{j} \cancel{+4\vec{j}\vec{k}}^{0} \cancel{-6\vec{k}\vec{i}}^{0} \cancel{+3\vec{k}\vec{j}}^{0} - 2\vec{k}\vec{k}}{21}$$

$$= \frac{12(1) - 6(1) - 2(1)}{21} = \frac{4}{21}$$

$$\cos\theta = 0.1905 \Rightarrow \boxed{\theta = 79°}$$

5) Show the given vectors are perpendicular to each other.

$\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$

$$\vec{a}.\vec{b} = \|\vec{a}\|.\|\vec{b}\|\cos\theta$$

$\vec{b} = \vec{i} + 2\vec{k}$

$$\begin{aligned}\vec{a}.\vec{b} &= (2\vec{i}+3\vec{j}-\vec{k})(\vec{i}+2\vec{k}) \\ &= 2\vec{i}.\vec{i} \cancel{+4\vec{i}\vec{k}}^{0} \cancel{+3\vec{j}\vec{i}}^{0} \cancel{+6\vec{j}\vec{k}}^{0} \cancel{-\vec{k}.\vec{i}}^{0} - 2\vec{k}\vec{k} \\ &= 2.(1) - 2(1) \\ &= 0\end{aligned}$$

$$0 = \cos\theta \Rightarrow \boxed{\theta = 90°}$$

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Eigen and Eigen Vectors

Let A be an $n \times n$ square matrix and X a vector with n components. In this case

$$Y = AX$$

product can be considered as a linear transformation from n-dimensional space to itself.

$$AX = \lambda X$$

The problem of finding λ scalars and different X vectors, is known as the eigen-eigen vector problem. In general, λ values and X vectors can have complex elements. However, here we will focus on examples containing real numbers. Eigen-eigen vector problems are always encountered in the solutions of vibration and balance problems in physics and engineering, and differential equations.

Characteristic Determinant, ~~Polynomial~~ Polynomial and Equality

Characteristic determinant:

If A is $n \times n$ matrix, characteristic determinant of A matrix is defined as $\det(\lambda I - A)$

Characteristic Polynomial:

The expansion of the characteristic determinant is an n^{th} polynomial in terms of λ, and is known as the characteristic polynomial

Characteristic equality:

$\det(\lambda I - A) = 0$ equality is known as the characteristic equality of the A matrix.

Characteristic polynomial of A matrix $(n \times n)$:

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \dots c_n$$

$$Ax = \lambda X$$

$$(\lambda x - Ax) = 0 \qquad \lambda = \lambda I_n$$

$$(\lambda I_n)x - Ax = 0$$

$$(\lambda I_n - A)x = 0$$

$$\det(\lambda I_n - A) = 0$$

eşitliğini sağlayan λ değerler ~~ise~~ A matrisinin özdeğerleridir.

Exp: i) Obtain the characteristic determinant of A matrix.

$$A=\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$$

$$\det(\lambda I-A)=\det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}\right)=\det\begin{bmatrix}\lambda-3 & -2\\ 1 & \lambda\end{bmatrix}$$

ii) Obtain the characteristic polynomial of A matrix

$$\det(\lambda I-A)=\begin{bmatrix}\lambda-3 & -2\\ 1 & \lambda\end{bmatrix}=\begin{vmatrix}\lambda-3 & -2\\ 1 & \lambda\end{vmatrix}=\lambda^2-3\lambda+2$$

iii) Obtain the characteristic equality of A matrix

$$\det(\lambda I-A)=\det\begin{bmatrix}\lambda-3 & -2\\ 1 & \lambda\end{bmatrix}=\lambda^2-3\lambda+2=0$$

$\lambda_1=1$ and $\lambda_2=2$ which are eigen values of A matrix.

Exp: Obtain the eigen values of A matrix.

$$A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$$

$$\lambda I-A=\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}-\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$$

$$(\lambda I-A)=\begin{bmatrix}\lambda+1 & -4 & 2\\ 3 & \lambda-4 & 0\\ 3 & -1 & \lambda-3\end{bmatrix}$$

$$\det(\lambda I-A)=\begin{vmatrix}\lambda+1 & -4 & 2\\ 3 & \lambda-4 & 0\\ 3 & -1 & \lambda-3\end{vmatrix}=0$$

$$=(\lambda+1)\begin{vmatrix}\lambda-4 & 0\\ -1 & \lambda-3\end{vmatrix}+(-1).(-4)\begin{vmatrix}3 & 0\\ 3 & \lambda-3\end{vmatrix}+2\begin{vmatrix}3 & \lambda-4\\ 3 & -1\end{vmatrix}$$

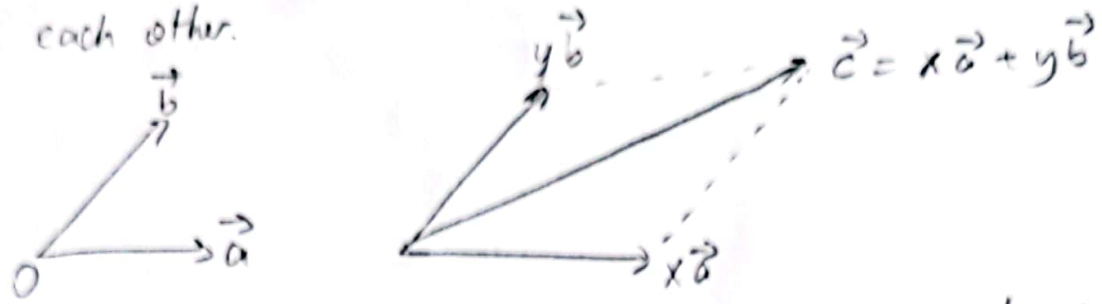
$$=(\lambda+1)\left[(\lambda-4)(\lambda-3)-0\right]+4\left[3(\lambda-3)-0\right]+2\left[-3-3(\lambda-4)\right]$$

$$(\lambda-3)(\lambda^2-3\lambda+2)=0$$

$$(\lambda-3)(\lambda-2)(\lambda-1)=0 \qquad \lambda_1=3,\ \lambda_2=2,\ \lambda_3=1$$

Basis

consider $\vec{a}$ and $\vec{b}$ vectors whose directions are different from each other.



For any vector $\vec{c} \neq 0$ of this plane, there can be x and y numbers, at least one of which is non-zero, satisfying the $\vec{c} = x\vec{a} + y\vec{b}$ equation. Because a parallelogram with a diagonal can be constructed on its vectors.

In other words, in the plane, two non-parallel vectors constitute a basis, and all vectors of the plane can be expressed as a linear combination of these base vectors.

Definition: If V is any vector space and $S = \{v_1, v_2, \dots v_n\}$ is a set of vectors in this space, then S is a basis for the V space if the following conditions are provided.

a) S is linear independent

b) S spans V.

Theorem: If $S = \{v_1, v_2 \dots v_n\}$ is a basis for V vector space, each v vectors in V is expressed as:

$$v = c_1 v_1 + c_2 v_2 + \dots c_n v_n$$

Exp: Find the solution of the homogenous linear equation system and specify a basis for the solution space.

$$\begin{aligned}
2x_1 + 2x_2 - x_3 + \quad\quad\quad x_5 &= 0 \\
-x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\
x_1 + x_2 - 2x_3 \quad\quad\quad - x_5 &= 0 \\
x_3 + x_4 + x_5 &= 0
\end{aligned}$$

(2a)

Solution: In the augmented matrix form:

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{If row operations applied on the matrix,}}$$ the solution is obtained as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ spans the solution space.

In addition, $\{v_1, v_2\}$ is a basis due to being linear independent and the solution space is 2-dimension.

Exp: Find the eigen values and eigen vectors of the matrix A.

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$$

Solution: $\det(\lambda I - A) = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} \right| = \begin{vmatrix} \lambda-3 & -1 \\ 2 & \lambda \end{vmatrix} = \lambda(\lambda-3)+2=0$

$\lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 2$ and $\lambda_2 = 1$ (eigen values)

i) In order to obtain the eigen vector corresponding to $\lambda_2 = 1$, it is used:

$(\lambda I - A)X = 0$

$\left(1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\right) X = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{array}{l} -2x_1 - x_2 = 0 \\ 2x_1 + x_2 = 0 \end{array}\right\} x_2 = -2x_1 \quad x_1 = t \Rightarrow x_2 = -2t \Rightarrow \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

is the eigen vector for $\lambda_2 = 1$ eigen value i.) In order to obtain the eigenvector corresponding to $\lambda_1 = 2$:

$(\lambda I - A)X = 0$

$$\left(2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}3 & 1\\-2 & 0\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \begin{bmatrix}-1 & -1\\2 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left.\begin{array}{l}-x_1 - x_2 = 0\\2x_1 + 2x_2 = 0\end{array}\right\} x_2 = -x_1 \qquad x_1 = t \Rightarrow x_2 = -t$$

$\begin{bmatrix}t\\-t\end{bmatrix} = t\begin{bmatrix}1\\-1\end{bmatrix}$ is the eigenvector for $\lambda_1 = 2$ eigen value.

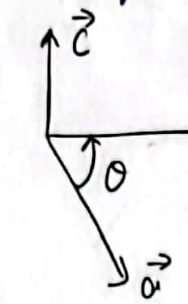
Homework: Find the eigen values and eigen vectors of the matrix given in the following.

$$A = \begin{bmatrix}1 & 0 & 2\\0 & -1 & -2\\-2 & -2 & 0\end{bmatrix}$$

Answer:

Eigenvalue	eigenvector
$\lambda_1 = 0 \rightarrow$	$\begin{bmatrix}2\\-2\\1\end{bmatrix}$
$\lambda_2 = 3 \rightarrow$	$\begin{bmatrix}2\\1\\-2\end{bmatrix}$
$\lambda_3 = -3 \rightarrow$	$\begin{bmatrix}1\\2\\2\end{bmatrix}$

Multiplication of Vectors. (vectorial)



There are $\vec a$ and $\vec b$ with angle of θ between them

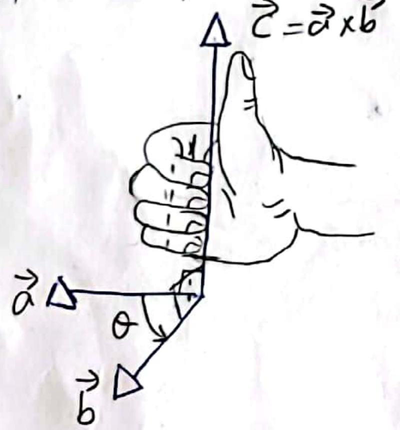
$$\vec c = \vec a \times \vec b = |\vec a||\vec b|\sin\theta$$

(Yanlış Kontrol et)

$\frac{i\,j\,k}{5} = 15$

Right Hand Rule

$\vec c = \vec a \times \vec b$, $\vec c$ is perpendicular with both $\vec a$ and $\vec b$.
The direction of $\vec c$ can be defined by using right hand rule.
If the right hand is held with four fingers pointing in the direction of rotation, it indicates the direction of the tumb.



Rules about Multiplication of ~~Vectorial~~ Vectors vectorally

1- $\vec a \times \vec b = -\vec b \times \vec a$

2- $\vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$

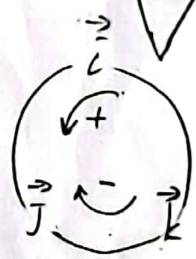
3- $m(\vec a \times \vec b) = (m\vec a)\times \vec b = \vec a \times (m\vec b) = (\vec a \times \vec b)m$

4- $\vec i \times \vec i = \vec j \times \vec j = \vec k \times \vec k = 0$

$\vec i \times \vec j = \vec k$ $\quad \vec j \times \vec k = \vec i$ $\quad \vec k \times \vec i = \vec j$

5- $\vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

6- $\vec a \times \vec b = 0 \Rightarrow \vec a \parallel \vec b$



Exp: $\left.\begin{array}{l} \vec a = \vec i + 2\vec j - 2\vec k \\ \vec b = 3\vec i + \vec k \end{array}\right\}$ $\vec c = \vec a \times \vec b = ?$

$$\vec c = \vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec i \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \vec j \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \vec k \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$\vec c = 2\vec i - 7\vec j - 6\vec k$$

2. method:

$$\vec c = \vec a \times \vec b = (\vec i + 2\vec j - 2\vec k)\times(3\vec i + \vec k)$$
$$= (\vec i \times 3\vec i) + (\vec i \times \vec k) + (2\vec j \times 3\vec i) + (2\vec j \times \vec k) + (-2\vec k \times 3\vec i) + (-2\vec k \times \vec k)$$
$$(0) + (-\vec j) + (6(-\vec k)) + (2(\vec i)) + (-6(\vec j)) - 2(0)$$
$$= -\vec j - 6\vec k + 2\vec i - 6\vec j$$
$$= 2\vec i - 7\vec j - 6\vec k$$

Properties of Eigen values and Eigen Vectors

Independence of Eigen Vectors

If eigen values $(\lambda_1, \lambda_2 \ldots \lambda_n)$ of a $n\times n$ A matrix and the bases of $B_1, B_2, \ldots B_n$ corresponding to the eigen vectors, the union of $B_1 \cup B_2 \cup \ldots \cup B_n$ is a linearly independent set.

Exp: It was obtain from the homework last week, $\lambda_1 = 0$ $\lambda_2 = 3$ and $\lambda_3 = -3$ and the bases are

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

If the union of these base is written as a matrix, and the determinant of the matrix is equal to zero, these column vectors are linearly independent.

$$\begin{vmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 2(2+4) - 2(-4-2) + 1(4-1)$$
$$= 12 + 12 + 3 = \underline{\underline{27 \neq 0}}$$

Properties of Eigen Values

If A is a $n\times n$ matrix and λ is a eigen value of the A matrix, $c\lambda$ is an eigen value of cA matrix. (c is a real number different from 0)

Eigen values of Triangular Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \text{ or } \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

If we consider upper-triangular matrix:

$$\det(\lambda I - A) = \det\begin{vmatrix} (\lambda - a_{11}) & -a_{12} & -a_{13} & -a_{14} \\ 0 & (\lambda - a_{22}) & -a_{23} & -a_{24} \\ 0 & 0 & (\lambda - a_{33}) & -a_{34} \\ 0 & 0 & 0 & (\lambda - a_{44}) \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$
$$= 0 \Rightarrow$$
$$\lambda = a_{11},\ \lambda = a_{22},\ \lambda = a_{33}$$
$$\lambda = a_{44}$$

Theorem: Eigen values of A matrix are diagonal elements of the matrix A, if A is $n\times n$ triangular matrix.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix} \quad \begin{matrix} \lambda = 3 \\ \lambda = 7 \\ \lambda = 1 \end{matrix}$$

Transformation the Matrices to Diagonal Form

Matrices that can be ~~converted~~ transformed to Diagonal

A square A matrix $(n \times n)$ can be transformed to diagonal form, if there is an inverse of P matrix by resulting $P^{-1}AP$ to be a diagonal matrix.

The sufficient condition to be transformed the A matrix to diagonal form is that eigen vectors of A matrix should be linearly independent.

If there is different eigen values, the eigen vectors corresponding to the eigen values are linearly independent. Therefore, If A matrix has different eigen values, the matrix can be transformed to diagonal form.

Exp: Determine whether the A matrix can be transformed to diagonal form, or not.

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0 \quad \lambda_1 = 3 \quad \lambda_2 = 2$$

Because λ_1 and λ_2 are different the A matrix can be transformed to diagonal.

~~Exp~~: ~~Exp~~ Steps of the methodology for transforming ~~the~~ a matrix to diagonal form.

Step-1: Obtain the eigen values $(\lambda_1, \lambda_2 .. \lambda_n)$

Step-2: Find n linear independent eigen vectors $(p_1, p_2 .. p_n)$

Step-3: Create a P matrix including $p_1, p_2 .. p_n$ vectors $P = [p_1, p_2 .. p_n]$

Step-4: Find P^{-1}.

Step-5: Find $D = P^{-1}AP$ diagonal matrix. The diagonal elements of this matrix are the eigen values $\lambda_1, \lambda_2 --- \lambda_n$.

Can the matrix given below be transformed to diagonal form? 3
If it can, transform.

$$A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$$

$$|\lambda I-A|=\begin{vmatrix}\lambda&0&+2\\-1&\lambda-2&-1\\-1&0&\lambda-3\end{vmatrix}=\lambda^3-5\lambda^2+8\lambda-4=(\lambda-1)(\lambda-2)^2=0$$

$$\lambda_1=1\quad \underbrace{\lambda_2=2\quad \lambda_3=2}$$

They are same, can the matrix be transformed to diagonal?

i) $(\lambda I-A)x=0$ for $\lambda=2$

$$\begin{bmatrix}2&0&2\\-1&0&-1\\-1&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&0&2&0\\-1&0&-1&0\\-1&0&-1&0\end{array}\right]\xrightarrow{\frac{1}{2}r_1\to r_1}\left[\begin{array}{ccc|c}1&0&1&0\\-1&0&-1&0\\-1&0&-1&0\end{array}\right]\xrightarrow[r_3+r_1\to r_3]{r_2+r_1\to r_2}\begin{array}{cccc}x_1&x_2&x_3&b\\ \end{array}\left[\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$\text{rank}(A)=1 \quad n=3 \quad n-r=3-1=2$ there is 2 independent variables.
The dependent variable is x_1.

$x_1+x_3=0 \quad x_1=-x_3 \quad x_2=s \quad x_3=t$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-t\\s\\t\end{bmatrix}=\begin{bmatrix}-t\\0\\t\end{bmatrix}+\begin{bmatrix}0\\s\\0\end{bmatrix}=t\begin{bmatrix}-1\\0\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

the linearly independent eigen vectors are $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$

ii) $(\lambda I-A)x=0$ for $\lambda=1$

$$\begin{bmatrix}1&0&2\\-1&-1&-1\\-1&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&0&2&0\\-1&-1&-1&0\\-1&0&-2&0\end{array}\right]\xrightarrow[r_3+r_1\to r_3]{r_2+r_1\to r_2}\left[\begin{array}{ccc|c}1&0&2&0\\0&-1&1&0\\0&0&0&0\end{array}\right]\xrightarrow{-r_2\to r_2}\begin{array}{cccc}x_1&x_2&x_3&b\\ \end{array}\left[\begin{array}{ccc|c}1&0&2&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

$\text{rank}(A)=2$
$n-r=3-2=1$ independent variable (x_3)

$x_1 + 2x_3 = 0 \qquad x_1 = -2x_3$

$x_2 - x_3 = 0 \qquad x_2 = x_3$

$$x_3 = t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector for $\lambda = 1$

$$P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{array}{c} \begin{array}{ccc} \lambda=2 & \lambda=2 & \lambda=1 \end{array} \\ \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\det P = \begin{vmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.(1) - 2(-1) = -1 + 2 = 1 \neq 0$$

P_1, P_2 and P_3 are linearly independent, even if $\lambda_2 = \lambda_3 = 2$.

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \text{ can be found.}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the diagonal form of A matrix

if P matrix is ordered as

$$P = \begin{array}{c} \begin{array}{ccc} \lambda=2 & \lambda=1 & \lambda=2 \end{array} \\ \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Homework:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{bmatrix}$$

Transform the A matrix to diagonal form

Answer:

$$D = P^{-1}AP = \begin{array}{c} \begin{array}{ccc} \lambda=0 & \lambda=3 & \lambda=-3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \end{array}$$